Important Rules

Important rules on trigonometric functions

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sin 2x = 2 \sin x \cos x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cos 2x = \cos^2 x - \sin^2 x$
$\cos x \cos y = [\cos(x-y) + \cos(x+y)]/2$	$\cos^2 x + \sin^2 x = 1$
$\sin x \sin y = [\cos(x-y) - \cos(x+y)]/2$	$\cos^2 x = \frac{(1+\cos 2x)}{2}$
$\sin x \cos y = [\sin(x-y) + \sin(x+y)]/2$	$\sin^2 x = \frac{(1-\cos 2x)}{2}$
$\sin x = \cos(90-\theta)$	$\tan x = \cot(90-\theta)$
$\cos x = \sin(90-\theta)$	$\cot x = \tan(90-\theta)$
$\tan^2 x = \sec^2 x - 1$	$\cot^2 x = \csc^2 x - 1$

Important rules on hyperbolic functions

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\cosh x + \sinh x = e^x$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
$\mathrm{csch}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$, $\mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$,
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
$\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = \cosh^2 x + \sinh^2 x$
$\cosh^2 x = \frac{(1+\cosh 2x)}{2}$, $\sinh^2 x = \frac{(\cosh 2x-1)}{2}$,
$\cosh^2 x - \sinh^2 x = 1$, $\tanh^2 x = 1 - \mathrm{sech}^2 x$, $\coth^2 x = 1 + \mathrm{csch}^2 x$

Important rules on logarithmic functions

1) $\log_a[xy] = \log_a[x] + \log_a[y]$
2) $\log_a[x/y] = \log_a[x] - \log_a[y]$
3) $\log_a[x^y] = y\log_a[x]$
4) $\log_a[x] = \ln x / \ln a$

Important rules on derivatives

Function	First derivative
$c^{f(x)}$	$\frac{d}{dx}[c^{f(x)}] = \frac{df}{dx} c^{f(x)} \ln c$
$\ln(f(x))$	$\frac{d}{dx}[\ln(f(x)] = \frac{df}{dx} / f(x)$
$\sin f(x)$	$\frac{d}{dx}[\sin f(x)] = \frac{df}{dx}[\cos f(x)]$
$\cos f(x)$	$\frac{d}{dx}[\cos f(x)] = -\frac{df}{dx}[\sin f(x)]$
$\tan f(x)$	$\frac{d}{dx}[\tan f(x)] = \frac{df}{dx} \sec^2 f(x)$
$\sec f(x)$	$\frac{d}{dx}[\sec f(x)] = \frac{df}{dx} \sec f(x) \tan f(x)$
$\mathrm{cosec}\, f(x)$	$\frac{d}{dx}[\mathrm{cosec}\, f(x)] = -\frac{df}{dx} \mathrm{cosec}\, f(x) \cot f(x)$

cot f(x)	$\frac{d}{dx}[\cot f(x)] = -\frac{df}{dx} \operatorname{cosec}^2 f(x)$
arccosf(x)	$\frac{d}{dx}[\arccos f(x)] = -\frac{1}{\sqrt{1-[f(x)]^2}} \frac{df}{dx}$
arcsin f(x)	$\frac{d}{dx}[\arcsin f(x)] = \frac{1}{\sqrt{1-[f(x)]^2}} \frac{df}{dx}$
arctan f(x)	$\frac{d}{dx}[\arctan f(x)] = \frac{1}{1+[f(x)]^2} \frac{df}{dx}$
arccot f(x)	$\frac{d}{dx}[\operatorname{arccot} f(x)] = -\frac{1}{1+[f(x)]^2} \frac{df}{dx}$
arcsec f(x)	$\frac{d}{dx}[\operatorname{arcsec} f(x)] = \frac{1}{[f(x)]\sqrt{[f(x)]^2-1}} \frac{df}{dx}$
arccsc f(x)	$\frac{d}{dx}[\operatorname{arccosec} f(x)] = -\frac{1}{[f(x)]\sqrt{[f(x)]^2-1}} \frac{df}{dx}$
cosh f(x)	$\frac{d}{dx}[\cosh f(x)] = \frac{df}{dx}[\sinh f(x)]$
sinh f(x)	$\frac{d}{dx}[\sinh f(x)] = \frac{df}{dx}[\cosh f(x)]$
tanh f(x)	$\frac{d}{dx}[\tanh f(x)] = \frac{df}{dx} \operatorname{sech}^2 f(x)$

sech f(x)	$\frac{d}{dx}[\operatorname{sech} f(x)] = -\frac{df}{dx} \operatorname{sech} f(x) \tanh f(x)$
csch f(x)	$\frac{d}{dx}[\operatorname{csch} f(x)] = -\frac{df}{dx} \operatorname{csch} f(x) \coth f(x)$
coth f(x)	$\frac{d}{dx}[\coth f(x)] = -\frac{df}{dx} \operatorname{cosech}^2 f(x)$
$\cosh^{-1} f(x)$	$\frac{d}{dx}[\cosh^{-1} f(x)] = \frac{1}{\sqrt{[f(x)]^2 - 1}} \frac{df}{dx}$
$\sinh^{-1} f(x)$	$\frac{d}{dx}[\sinh^{-1} f(x)] = \frac{1}{\sqrt{1 + [f(x)]^2}} \frac{df}{dx}$
$\tanh^{-1} f(x)$	$\frac{d}{dx}[\tanh^{-1} f(x)] = \frac{1}{1 - [f(x)]^2} \frac{df}{dx}$
$\coth^{-1} f(x)$	$\frac{d}{dx}[\coth^{-1} f(x)] = \frac{1}{1 - [f(x)]^2} \frac{df}{dx}$
$\operatorname{sech}^{-1} f(x)$	$\frac{d}{dx}[\operatorname{sech}^{-1} f(x)] = -\frac{1}{f(x)\sqrt{1 - [f(x)]^2}} \frac{df}{dx}$
$\operatorname{csch}^{-1} f(x)$	$\frac{d}{dx}[\operatorname{csch}^{-1} f(x)] = -\frac{1}{f(x)\sqrt{1 + [f(x)]^2}} \frac{df}{dx}$

Table of important integrals

$\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$
$\int u^{-1} \, du = \int \frac{1}{u} \, du = \ln \lvert u \rvert + C$
$\int e^u \, du = e^u + C$
$\int a^u \, du = \frac{1}{\ln a} a^u + C$
$\int \sin u \, du = -\cos u + C$
$\int \cos u \, du = \sin u + C$
$\int \sec^2 u \, du = \tan u + C$
$\int \sec u \tan u \, du = \sec u + C$
$\int \csc^2 u \, du = -\cot u + C$

$\int \csc u \cot u \, du = -\csc u + C$
$\int \tan u \, du = \ln \lvert \sec u \rvert + C = -\ln \lvert \cos u \rvert + C$
$\int \sec u \, du = \ln \lvert \sec u + \tan u \rvert + C$
$\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 - u^2} \, du = \frac{1}{2a} \ln \left\lvert \frac{a+u}{a-u} \right\rvert + C$
$\int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1 - u^2} + C$
$\int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1 - u^2} + C$
$\int \tan^{-1} u \, du = u \tan^{-1} u - \frac{1}{2} \ln\left(u^2 + 1\right) + C$

$\int \cot^{-1} u\, du = u \cot^{-1} u + \frac{1}{2} \ln\left(u^2 + 1\right) + C$
$\int \sec^{-1} u\, du = u \sec^{-1} u - \ln\left(u + \sqrt{u^2 - 1}\right) + C$
$\int \csc^{-1} u\, du = u \csc^{-1} u + \ln\left(u + \sqrt{u^2 - 1}\right) + C$
$\int e^{au}\, du = \frac{1}{a} e^{au} + C$
$\int u e^{au}\, du = \frac{1}{a^2} (au - 1) e^{au} + C$
$\int u^2 e^{au}\, du = \frac{1}{a^3} \left(a^2 u^2 - 2au + 2\right) e^{au} + C$
$\int u^2 e^{au}\, du = \frac{1}{a^3} \left(a^2 u^2 - 2au + 2\right) e^{au} + C$
$\int e^{au} \sin bu\, du = \frac{1}{a^2 + b^2} e^{au} (a \sin bu - b \cos bu) + C$
$\int e^{au} \cos bu\, du = \frac{1}{a^2 + b^2} e^{au} (a \cos bu + b \sin bu) + C$
$\int \ln u\, du = u \ln u - u + C$

$\int af(u)\,du = a\int f(u)\,du$
$\int [f(u)+g(u)]\,du = \int f(u)\,du + \int g(u)\,du$
$\int [f(u)-g(u)]\,du = \int f(u)\,du - \int g(u)\,du$
$\int [af(u)+bg(u)]\,du = a\int f(u)\,du + b\int g(u)\,du$
$\int u\,dv = uv - \int v\,du$ (Integration by parts)
$\int_a^b f(u)\,du = -\int_b^a f(u)\,du$ (Definite integral)
$\int_a^c f(u)\,du = \int_a^b f(u)\,du + \int_b^c f(u)\,du$ (Definite integral)

Chapter 5

Ordinary Differential equations

5.1 Introduction

In mathematics, an **ordinary differential equation** or **O.D.E.** is a differential equation containing a function or functions of one independent variable and its derivatives. The term "ordinary" is used in contrast with the term partial differential equation which may be with respect to more than one independent variable.

Linear differential equations, which have solutions that can be added and multiplied by coefficients, are well-defined and understood and exact closed-form solutions are obtained. By contrast, ODEs that lack additive solutions are nonlinear, and solving them is far more intricate, as one can rarely represent them by elementary functions in closed form: Instead, exact and analytic solutions of ODEs are in series or integral form. Graphical and numerical methods applied by hand or by computer, may approximate solutions of ODEs and perhaps yield useful information, often sufficing in the absence of exact, analytic solutions.

An ordinary differential equation (frequently called an "ODE," "diff eq," or "diffy Q") is an equality involving a function and its derivatives. An ODE of order n is an equation of the form $F(x,y,y`,y``,\dots,y^{(n)}) = 0$, where y is a function of x, y` is the first derivative with respect to x, y`` is the second derivative with respect to x,….$y^{(n)}$ is the n^{th} derivative with respect to x.

The following definitions must be studied carefully

Differential equation: A differential equation is an equation containing derivatives of a dependent variable with respect to one or more or independent variables. The following are typical examples:

$$\frac{d^2y}{dx^2} = 6x,$$

$$x\frac{dy}{dx} + 3y = y^3,$$

$$\left[\frac{d^2y}{dx^2}\right]^2 + 6\left[\frac{dy}{dx}\right]^3 = y.$$

Ordinary differential equation: A differential equation containing a single independent variable. The derivatives occurring in the equation are ordinary derivatives.

Partial differential equation: A differential equation containing two or more independent variables. The derivatives occurring in the equation are partial derivatives.

Order of a differential equation: The order of the highest ordered derivative occurring in the equation.

For the above examples, the first and the third differential equations are of order 2 and the second differential equation is of order 1.

Degree of a differential equation: In general, the degree of the highest ordered derivative occurring in the equation. However, not every differential equation has a degree. If the derivatives occur within radicals or fractions the equation may not have a degree. If the equation can be rationalized and cleared of fractions with regard to all derivatives present, then its degree is the degree of the highest ordered derivative occurring in the equation.

For the above examples, the first and the second differential equations are of degree 1 and the third differential equation is of degree 2.

Linear differential equation: A linear differential equation is an equation of the form

$$a_0 \frac{dy^n}{d^n x} + a_1 \frac{dy^{n-1}}{d^{n-1} x} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

where the $a_i(x)$ are functions of x only. It is an equation in which each term is of first degree in the dependent variable and its derivatives.

Solutions of differential equations: A solution of a differential equation is any relation, free of derivatives, between the variables involved that reduces the differential equation to an identity. The solution may take the form of the dependent variable being expressed explicitly as a function of the independent variable (or variables) as in $y = f(x)$ or implicitly as in a relation of the type $f(x, y) = 0$.

5.2 First order ordinary D.E.

In this section we will look at solving first order differential equations. The most general first order differential equation can be written as,

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

As we will see in this chapter there is no general formula for the solution to (1). What we will do instead is look at several special cases and see how to solve those. We will also look at some of the theory behind first order differential equations as well as some applications of first order differential equations. Below is a list of the topics discussed in this chapter.

Separable Equations: Identifying and solving separable first order differential equations. We'll also start looking at finding the interval of validity from the solution to a differential equation.

Substitutions: there are many types of differential equations using substitution methods, such as homogenous, non homogenous, same slope equations.

Exact Equations: Identifying and solving exact differential equations. We'll do a few more intervals of validity problems here as well.

Linear Equations: Identifying and solving linear first order differential equations.

Bernoulli Differential Equations : In this section we'll see how to solve the Bernoulli Differential Equation.

This section will also introduce the idea of using a substitution to help us solve differential equations and now we will discuss each method used in solving first order ordinary differential equations.

5.2.1 Separable Equations

The differential equation of the form $\frac{dy}{dx}=f(x,y)$ is called separable**,** if

$$f(x,y) = h(x)\, g(y) \qquad (2)$$

That is, $\frac{dy}{dx}=h(x)\, g(y)$. In order to solve it, perform the following steps:

- Rewrite the equation $\frac{dy}{dx}=h(x)\, g(y)$ as $\frac{dy}{g(y)}=h(x)\,dx$
- Integrate $\frac{dy}{g(y)}=h(x)\,dx$ to obtain G(y)= H(x) + C
- If you are given an IVP, use the initial condition to find the particular solution.

Example 1 : Find the particular solution of $\frac{dy}{dx}=\frac{y^2-1}{x}$, y(1) = 2

Solution: Rewrite the equation as $\frac{dy}{y^2-1}=\frac{dx}{x}$, then using the techniques of integration of rational functions with the aid of the partial fraction , we get $\int \frac{dy}{y^2-1} = \frac{1}{2}\ln\frac{|y-1|}{|y+1|}$ which implies the solutions to the given differential equation is $\frac{1}{2}\ln\frac{|y-1|}{|y+1|} = \ln|x| + C$. We need to find the constant C, so that if we plug in the condition y=2 when x=1, we get C = $\frac{1}{2}\ln(\frac{1}{3})$, note that this solution is given in an implicit form. You may be asked to rewrite it in an explicit one. For example, in this case, we have $y = \frac{3+x^2}{3-x^2}$.

Example 2 : Find solution to $\frac{dy}{dt} = 1 + \frac{1}{y^2}$

Solution: Rewrite the equation as $\frac{y^2dy}{y^2+1} = dt$, then integrate such that:

$$\int \frac{y^2dy}{y^2+1} = \int \frac{(y^2+1-1)dy}{y^2+1} = \int (1 - \frac{1}{y^2+1})dy = \int dt + C$$

which implies the solutions to the given differential equation is $y - \tan^{-1}(y) = t + C$

Example 3 : Solve the initial value problem

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2y^2, \quad y(0) = 1$$

Solution: Rewrite the equation as $\frac{dy}{y^2+1} = (t^2+1)dt$, then integrate such that $\int \frac{dy}{y^2+1} = \int (t^2+1)dt + C$ which implies the solutions to the given D.E. is $\tan^{-1}y = \frac{t^3}{3} + t + C$, but for t = 0, y = 1, $C = \tan^{-1}1 = \frac{\pi}{4}$, thus $\tan^{-1}y = \frac{t^3}{3} + t + \frac{\pi}{4}$.

5.2.2 Homogeneous Equations

The differential equation $\frac{dy}{dx} = f(x,y)$ is homogeneous if the function f(x,y) is homogeneous such that :

$$f(tx,ty) = f(x,y) \quad \text{for any number } t. \tag{3}$$

Example 4: Check that the functions are homogeneous.

$$f(x,y)=\ln(\frac{-3x^2y}{x^3+4xy^2}), \quad g(x,y)=(\frac{xy}{x^2+y^2})$$

Solution: $f(tx,ty)=\ln(\frac{-3(tx)^2 ty}{(tx)^3+4tx(ty)^2})=f(x,y)$,

$$g(tx,ty)=(\frac{tx(ty)}{(tx)^2+(ty)^2})= g(x,y)$$

In order to solve this type of equation we make use of a substitution $v = \frac{y}{x}$. Let us summarize the steps to follow:

- Recognize that your equation is an homogeneous equation; that is, you need to check that f(tx,ty)= f(x,y), meaning that f(tx,ty) is independent of the variable t; then write out the substitution v =y/x; hence put y = vx and dy = vdx +xdv in the equation.
- Through easy differentiation, find the new equation satisfied by the new function v.
- Solve the new equation (which is always separable) to find v.
- Go to the old function y through the substitution y= vx.

- If you have an I.V.P., use the initial condition to find the particular solution.

Example 5: Find solution to $\frac{dy}{dx}=\frac{-2x+5y}{2x+y}$

Solution: Follow these steps:

- Since $\frac{dy}{dx}=\frac{-2x+5y}{2x+y}$ is homogeneous;
- Put y = vx , hence dy= v dx + x dv, therefore

$$\frac{vdx\ +xdv}{dx}=\frac{-2x+5vx}{2x+vx}=\frac{-2+5v}{2+v},$$

Hence

$$\frac{dx}{x}+\frac{(2+v)dv}{v^2-3v+2}=0$$

- This is a separable equation , we can solve it using partial fraction, so that

$$\int\frac{dx}{x}+\int(\frac{4}{v-2}-\frac{3}{v-1})dv=C$$

- Integrate, we get its solution such that

lnx = 4ln(v-2) - 3 ln(v-1) + C, but v = y/x , therefore 2lnx = 4 ln(y-2x) - 3 ln(y-x) + C.

Example 6: Find solution to $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$

Solution: Since $\frac{xy}{x^2+y^2}$ is homogeneous, put y = vx, and dy = vdx + xdv in the above equation, we get $\frac{vdx+xdv}{dx}=\frac{x(vx)}{x^2+(vx)^2}=\frac{v}{1+v^2}$, it is a separable equation such that: $\frac{dx}{x}+\frac{(1+v^2)dv}{v^3}=0$, integrate this equation, such that ln x-$\frac{1}{2v^2}$ +ln v= C, but y = vx , therefore ln y -$\frac{x^2}{2y^2}$= C.

5.2.3 Same Slope Equations

The differential equation of the form $\frac{dy}{dx}=f(x,y)$ is called same slope equation if

$$f(x,y) = \frac{ax+by+c}{px+qy+s} \text{ and } \frac{a}{P}=\frac{b}{q}=L \tag{4}$$

L(px + qy) = ax + by , s, c are constants of the straight lines. In order to solve this type of equation we make use of a substitution u= px + qy, therefore du = pdx + qdy, and ax + by = Lu, hence the differential equation will be in the form $\frac{du - pdx}{qdx} = \frac{Lu + c}{u + s}$, it can be expressed in the form $\frac{(u+s)du}{p(u+s) + q(Lu+c)} = dx$ which is separable equation.

Example 7 : Find solution to $\frac{dy}{dx} = \frac{2x + 3y + 7}{4x + 6y + 28}$

Solution:

Since $2x + 3y + 7$, $4x + 6y + 28$ are two first degree expressions with the same slope, therefore $\frac{dy}{dx} = \frac{2x + 3y + 7}{4x + 6y + 28}$ is called same slope equation. To solve this differential equation, we have to follow these steps:

- Put u = $2x + 3y \Rightarrow$ du = 2dx+3dy , so $dy = \frac{1}{3}(du - 2dx)$
- Substitute in the differential equation, we get

$$\frac{(du - 2dx)}{3dx} = \frac{u + 7}{2u + 28},$$

Therefore

$$dx = \frac{(2u+28)du}{(7u+77)} = \frac{2}{7}(\frac{u+14}{u+11})du\,,$$

Thus

$$dx = \frac{2}{7}(\frac{u+14}{u+11})du = \frac{2}{7}(\frac{u+11+3}{u+11})du = \frac{2}{7}(1+\frac{3}{u+11})du$$

Integrate the above equation such that

$$\int dx = \int \frac{2}{7}(1+\frac{3}{u+11})du + C$$

Therefore

$$x = \frac{2}{7}[u + 3\ln(u+11)] + C, \text{ but } u = 2x + 3y$$

hence the solution of differential equation is

$$x = \frac{2}{7}[2x + 3y + 3\ln(2x + 3y + 11)] + C$$

5.2.4 Non Homogeneous Equations

The differential equation of the form $\frac{dy}{dx} = f(x,y)$ is called non homogeneous equation if f(x,y) = $\frac{ax+by+c}{px+qy+s}$, where ax + by + c = 0, px + qy + s=0 are not parallel and not homogeneous. To solve the differential equation , we have to follow the following steps:

- Solve the two lines ax + by + c = 0, px + qy + s = 0 to get the point of intersection which is $(\frac{bs-qc}{aq-bp}, \frac{pc-as}{aq-bp})$,
- Put x = X + $\frac{bs-qc}{aq-bp}$, y = Y+$\frac{pc-as}{aq-bp}$, and dx=dX, dy=dY, so the differential equation is transformed into $\frac{dY}{dX} = \frac{aX+bY}{pX+qY}$ which become a homogeneous equation so that we can solve it by substitution Y= vX $\Rightarrow$ dy = vdX+Xdv, then the differential equation will be separable equation such that:

$$\frac{vdX+Xdv}{dX}=\frac{aX+bvX}{pX+qvX}=\frac{a+bv}{p+qv},$$

- Integrate the differential equation $\frac{dX}{X}=\frac{(p+qv)dv}{qv^2+bv+a}$
- Substitute $X = x - \frac{bs-qc}{aq-bp}$, $Y = y - \frac{pc-as}{aq-bp}$, and

$$v=\frac{Y}{X}=\frac{y(aq-bp)-(pc-as)}{x(aq-bp)-(bs-qc)}.$$

Example 8 : Find solution to $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$

Solution: Since $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$ is non homogeneous equation.

To solve this differential equation, we have to follow these steps

- We have to get the point of intersection between $x + y + 3 = 0$, $x - y + 1= 0$ which is (-2,-1),
- Put $x = X-2$, $y=Y-1$, $dx = dX$, $dy = dY$ in the above differential equation, then $\frac{dY}{dX}=\frac{X+Y}{X-Y}$, so it is a homogeneous equation,

- Put Y = vX, and dY = vdX + Xdv, therefore

$$\frac{vdX+Xdv}{dX}=\frac{X+vX}{X-vX}=\frac{1+v}{1-v}$$

- Integrate $\frac{dX}{X}=\frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v=\frac{Y}{X}=\frac{y+1}{x+2}$,

hence the solution of the differential equation is

$$\text{Ln}(x+2) = \tan^{-1}(\frac{y+1}{x+2}) - \frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2})+C$$

5.2.5 Exact and non exact Equations

All the techniques we have reviewed so far were not of a general nature since in each case the equations themselves were of a special form. So, we may ask, what to do for the general equation $\frac{dy}{dx}=f(x,y)$

Let us first rewrite the equation into M(x,y)dx + N(x,y)dy = 0
This equation will be called exact if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \text{ and non exact otherwise} \qquad (5)$$

The condition of exactness insures the existence of a function f(x,y) such that :

$$\frac{\partial f}{\partial x} = M(x,y),\ \frac{\partial f}{\partial y} = N(x,y) \tag{6}$$

When the equation is exact, we solve it using the following steps:

- Check that the equation is indeed exact;
- Write down the system $\frac{\partial f}{\partial x} = M(x,y),\ \frac{\partial f}{\partial y} = N(x,y)$
- Integrate either the first equation with respect of the variable x or the second with respect of the variable *y*. The choice of the equation to be integrated will depend on how easy the calculations are. Let us assume that the first equation was chosen, then we get

$$f(x,y) = \int M(x,y)dx + \phi(y)$$

The function $\phi(y)$ should be there, since in our integration, we assumed that the variable *y* is constant.

- Use the second equation of the system to find the derivative of $\phi(y)$. Indeed, we have

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(\int M(x,y)dx)+\phi'(y)=N(x,y)$$

which implies

$$\phi'(y)=N(x,y)-\frac{\partial}{\partial y}(\int M(x,y)dx)$$

Note that ϕ is a function of y only. Therefore, in the expression giving $\phi'(y)$, the variable x should disappear. Otherwise something went wrong!

- Integrate to find $\phi(y)$; then write down the function f(x,y)
- All the solutions are given by the implicit equation f(x,y)= C
- If you are given an I.V.P., plg in the initial condition to find the constant C.

You may ask, what do we do if the equation is not exact? In this case, one can try to find an integrating factor which makes the given differential equation exact.

<u>Integrating Factor Technique</u>

Assume that the equation M(x,y)dx + N(x,y)dy = 0 is **not exact**, that is $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. In this case we look for a function u(x,y) which makes the new equation u(x,y) M(x,y)dx + u(x,y) N(x,y)dy = 0, an exact one. The function u(x,y) (if it exists) is called the integrating factor. Note that u(x,y) satisfies the following equation:

$$\frac{\partial M}{\partial y}u + \frac{\partial u}{\partial y}M = \frac{\partial N}{\partial x}u + \frac{\partial u}{\partial x}N$$

This is not an ordinary differential equation since it involves more than one variable. This is what's called a partial differential equation. These types of equations are very difficult to solve, which explains why the determination of the integrating factor is extremely difficult except for the following two special cases:

Case 1: There exists an integrating factor u(x) function of x only. This happens if the expression

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} \qquad (7)$$

is a function of x only, that is, the variable y disappears from the expression. In this case, the function u is given by

$$u(x)=\exp(\int \frac{M_y - N_x}{N}\,dx)$$

Case 2: There exists an integrating factor u(y) function of y only. This happens if the expression

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} \qquad (8)$$

is a function of y only, that is, the variable x disappears from the expression. In this case, the function u is given by

$$u(x)=\exp(\int \frac{N_x - M_y}{M}\,dy)$$

Once the integrating factor is found, multiply the old equation by u to get a new one which is exact. Then you are left to use the previous technique to solve the new equation.

Advice: if you are not pressured by time, check that the new equation is in fact exact!

Let us summarize the above technique. Consider the equation

$$M(x,y)dx + N(x,y)dy = 0$$

If your equation is not given in this form you should rewrite it first.

Step 1: Check for exactness, by computing $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$, then compares them.

Step 2: Assume that the equation is not exact, then evaluate

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

If this expression is a function of x only, then go to step 3. Otherwise, evaluate

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

If this expression is a function of y only, then go to step 3. Otherwise, you can not solve the equation using the technique developed above!

Step 3: Find the integrating factor. We have two cases:

a) If the expression $\frac{M_y - N_x}{N}$ is a function of x only. Then an integrating factor is given by $u(x) = \exp(\int \frac{M_y - N_x}{N} dx)$,

b)If the expression $\frac{N_x - M_y}{M}$ is a function of y only, then an integrating factor is given by $u(x) = \exp(\int \frac{N_x - M_y}{M} dy)$

Step 4: Multiply the old equation by u, and, if you can, check that you have a new equation which is exact.

Step 5: Solve the new equation using the steps described in the previous section.

Example 9 : Find solution to $\frac{dy}{dx}=-\frac{2x+6xy^2}{3y^2+6x^2y}$

Solution:

Since the equation is $(2x+6xy^2)$dx+$(3y^2+6x^2y)$dy=0, M(x,y)= $2x+6xy^2$, N(x,y) = $3y^2+6x^2y$ & $\frac{\partial M}{\partial y}=12xy=\frac{\partial N}{\partial x}$, then the equation is exact, but $\frac{\partial f}{\partial x}=M(x,y)=2x+6xy^2$, thus f(x,y) = $x^2 + 3x^2y^2 + \phi(y)$, $\frac{\partial f}{\partial y}=6x^2y+\phi'(y)=3y^2+6x^2y$, from which we conclude $\phi'(y)=3y^2$, so $\phi(y)=y^3$, thus the solution of the D.E. is f(x,y) = $x^2 + 3x^2y^2 + y^3$.

The following example illustrates the use of the integrating factor technique:

Example 10 : Find the solution to $\frac{dy}{dx}=-\frac{3xy+y^2}{x^2+xy}$

Solution:

1) Rewrite the equation to get $(3xy+y^2)dx+(x^2+xy)dy=0$, hence $M(x,y)=3xy+y^2$ and $N(x,y)=x^2+xy$.

(2) We have $\frac{\partial M}{\partial y}=3x+2y$, $\frac{\partial N}{\partial x}=2x+y$, which clearly implies that the equation is not exact.

(3) Let us find an integrating factor. We have $\frac{M_y-N_x}{N}=\frac{1}{x}$.

Therefore, an integrating factor u(x) exists and is given by

$$u(x)=\exp\left(\int\frac{M_y-N_x}{N}dx\right)=\exp\left(\int\frac{1}{x}dx\right)=\exp(\ln x)=x$$

(4) The new equation is $(3x^2y+xy^2)dx+(x^3+x^2y)\,dy=0$, which is exact. (Check it!)

5.2.6 First Order Linear Equation

A first order linear differential equation has the following form:

$$\frac{dy}{dx}+p(x)y=q(x) \qquad (9)$$

The general solution is given by $y(x)=\dfrac{\int u(x)\,q(x)\,dx+C}{u(x)}$, where $u(x)=\exp(\int p(x)dx)$ is called the integrating factor. If an initial condition is given, use it to find the constant C. Here are some practical steps to follow:

1) If the D.E. is given as $a(x)\dfrac{dy}{dx}+b(x)y=c(x)$, rewrite it in the form $\dfrac{dy}{dx}+p(x)y=q(x)$, i.e. $p(x)=\dfrac{b(x)}{a(x)}, q(x)=\dfrac{c(x)}{a(x)}$,

2) Find the integrating factor $u(x)=\exp(\int p(x)dx)$,

3) Evaluate the integral $\int u(x)\,q(x)\,dx$,

4) Write the general solution $y(x)=\dfrac{\int u(x)\,q(x)\,dx+C}{u(x)}$,

5) If you are given an I.V.P., use the initial condition to find the constant C.

Example 11 : Find the solution to $y'+(\tan x)y=\cos^2 x$

Solution: Let us use the steps:

Step 1: There is no need for rewriting the differential equation. We have $p(x) = \tan x,\ q(x) = \cos^2 x$

Step 2: The integrating factor is

$$u(x) = \exp\left(\int \tan x\, dx\right) = \exp(-\ln \cos x) = \frac{1}{\cos x}$$

Step 3: Evaluate $\int u(x)\, q(x)\, dx$ such that:

$$\int u(x)\, q(x)\, dx = \int \frac{1}{\cos x}(\cos^2 x)\, dx = \int \cos x\, dx = \sin x$$

Step 4: The general solution is given by

$$y(x) = \frac{\int u(x)\, q(x)\, dx + C}{u(x)} = \frac{\sin x + C}{\sec x}$$

Step 5: In order to find the particular solution to the given I.V.P., we use the initial condition to find C. Indeed, we have y(0) = C = 2, therefore the solution is

$$y(x) = (\sin x + 2) \cos x.$$

5.2.7 Bernoulli Equation

A differential equation of Bernoulli type is written as

$$\frac{dy}{dx} + p(x)y = q(x)\, y^n \qquad (10)$$

This type of equation is solved via a substitution. Indeed, let $z = y^{1-n}$, then easy calculations give z\` = (1-n) y $^{-n}$ y\` which implies

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x) \tag{11}$$

This is a linear equation satisfied by the new variable z. Once it is solved, you will obtain the function $y = z^{1/1-n}$. Note that if n > 1, then we have to add the solution y = 0 to the solutions found via the technique described above. Let us summarize the steps to follow:

- Recognize that the differential equation is a Bernoulli equation. Then find the parameter n from the equation;
- Write out the substitution $z = y^{1-n}$;
- Through easy differentiation, find the new equation satisfied by the new variable z. You may want to remember the form of the new equation:

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

- Solve the new linear equation to find z;

- Go back to the old function *y* through the substitution $y = z^{1/1-n}$;
- If n > 1, add the solution y = 0 to the ones you obtained above;
- If you have an I.V.P., use the initial condition to find the particular solution.

Example 12 : Find the solution to $\frac{dy}{dx} = y + y^3$

Solution: Perform the following steps:

- We have a Bernoulli equation with n = 3;
- Let $z = y^{1-3} = y^{-2}$; $\frac{dz}{dx} = -2y^{-3}y'$
- The new equation satisfied by z is $\frac{dz}{dx} + 2z = -2$, P(x) = 2, q(x) = -2;
- This is a linear equation with integrating factor

$$u(x) = \exp(\int p(x)dx) = \exp(\int 2dx) = e^{2x} ,$$

also we have $\int u(x)\, q(x)\, dx = \int -2\, e^{2x}\, dx = -e^{2x}$, the general solution is given by

$$z(x) = \frac{\int u(x)\, q(x)\, dx + C}{u(x)} = \frac{-e^{2x} + C}{e^{2x}} = -1 + C\, e^{-2x},$$

but $y = z^{-1/2}$ which gives $y = \pm(-1 + C\, e^{-2x})^{-1/2}$

5.3 Problems

I) Solve the following differential equations

1) $(x+2y)\, dx + (2x-y+1)\, dy = 0$

2) $y` = \dfrac{2x - 3y + 9}{6y - 4x + 1}$

3) $(y + \ln(x))dx + (x+y^2)\, dy = 0$

4) $y` = y + \sin(x)$

5) $y` = \dfrac{-2x + 5y}{2x + y}$

6) $(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$

7) $y` + (\tan x)\, y = \cos^2 x$

8) $y` = y + y^3$

9) $xy` + y = -x^3$

10) $y` + y/x = -2$

11) $y` = e^{x-y}$

12) $y` = 1 + x + y + xy$

13) $xy` - \sin(x)/y = y\, Ln(x)$

14) $y` = \dfrac{\cos y - ye^x}{e^x + x\sin y}$

15) $y` = x^3 - (4/x)y$

16) $y` = \dfrac{xy + x^2}{y^2}$

17) $y` = (y/x) + \tan(y/x)$

18) $y` = x(2Lnx+1)/(\sin y + y\cos y)$

19) $y` = \sqrt{y-x}$ (Hint: put $\sqrt{y-x} = t$)

20) $y - xy` = a\,(y^2 + y`)$

21) $y` = y\,e^x - 2e^x + y - 2$

22) $y` = x\cos x\, y$

23) $y` = -2xy + 2x$

24) $y` = y \tan x - \cos x$

25) $y` = \frac{xy}{1+x^2} + \sqrt{\frac{1+x^2}{1-x^2}}$

26) $y` = (y/2x) - (xy)^3$

27) $\frac{x}{\sqrt{y^2+x^2}} + \frac{yy'}{\sqrt{y^2+x^2}} = 0$

28) $y^2 - xy + x^2 y` = 0$

29) $1 + y + (2y + 2y^2)y` = 0$

30) $\frac{1+y}{1+x} + y` = 0$

31) $1 - \frac{x}{x^2+y^2} - \frac{yy'}{x^2+y^2} = 0$

32) $x^2 + y/x + \mathrm{Ln}(xy)\, y` = 0$

33) $2x\, y^2 + 4x^3 + 2(x^2 + 1)\, y\, y` = 0$

34) $dy / dx - 2\, x\, y = x$

35) dy / dx + y / x = - 2 for x > 0

36) x dy / dx + y = - x3 for x > 0

37) dy / dx + y = 2x + 5

38) $y' = 3 e^{y} x^{2}$

39) y ' = sin x / (y cos y)

40) $y' = -9 x^2 y^2$

41) $y` = 1 + 1/ y^2$

42) $y` = y + y^3$

43) $\frac{dy}{dt} = \frac{2t}{1+t^2} y + \frac{2}{1+t^2}$, $y(0) = 0.4$

44) $(\cos^2 t \sin t)\frac{dy}{dt} = -\cos^3 t\, y + 1$, $y(\frac{\pi}{4}) = 0$

45) $2t\frac{dy}{dt} - y = t + 1$, $y(2) = 4$, $t > 0$

46) $x\,\dfrac{dy}{dx} + y = -\,x$

47) $\dfrac{dy}{dx} = 6xy^2$, $y(1) = 1/25$

48) $y' = \dfrac{3x^2+4x-4}{2y-4}$, $y(1) = 3$

49) $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

50) $\dfrac{dy}{dt} = e^{y-t}(\sec y)(1+t^2)$, $y(0) = 0$

51) $(2y + x^2 + 1)\dfrac{dy}{dx} - 9x^2 + 2xy = 0$, $y(0) = -3$

52) $2(3 - yx^2)\dfrac{dy}{dx} = 4 + 2xy^2$, $y(-1) = 8$

53) $\dfrac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0$, $y(5) = 0$

54) $3y^3e^{3xy} - 1 + (2y + 3xy^2)e^{3xy}y' = 0$, $y(0) = 1$

55) $y' + \dfrac{4}{x}y = x^3y^2$, $y(2) = -1$

56) $y' = 5y + e^{-2x} y^{-2}$, $y(0) = 2$

57) $6y' - 2y = xy^4$, $y(0) = -2$

58) $y' + \frac{y}{x} - \sqrt{y} = 0$, $y(1) = 0$

59) $\frac{dv}{dt} = 9.8 - 0.196v$

60) $\cos(x)\frac{dy}{dx} + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1$, $y(\frac{\pi}{4}) = 3\sqrt{2}$, $0 \le x < \frac{\pi}{2}$

61) $ty' + 2y = t^2 - t + 1$, $y(1) = ½$

62) $ty' - 2y = t^5 \sin(2t) - t^3 + 4t^4$, $y(\pi) = (3/2)\ \pi^4$

63) $xyy' + 4x^2 + y^2 = 0$, $y(2) = -7$

64) 2y` - y = 4 sin3t

65) 2x(y+1) dx – y dy = 0

66) $2(y-1)x^2$ dy + cscy dx = 0

67) $x \sec y\, dx + (1-6y^5)\, dy = 0$

68) $(x y^2 - x)\, dx + (x^2 y + y)\, dy = 0$

69) $x dy + y \, Ln x \, dx = 0$

70) $(2x-6y+3) - (x-3y+1)dy = 0$

71) $(x+2y-4)\, dx - (2x + y - 5)\, dy = 0$

72) $(x+2y-1)\, dx + 3(x+2y)\, dy = 0$

73) $e^{-y}(y`+1) = xe^x$ (Hint: put $x+y = u$)

74) $(x-y)dx + (x-4y)dy = 0$

75) $(x^2 - xy + y^2)\, dx - xy\, dy = 0$

76) $(y+x y^2)\, dx + (x + x^2 y)dy = 0$

77) $(x^2 - 2y^2)\, dx + xy\, dy = 0$

78) $xy\, dx + (x^2 + y^2)\, dy = 0$

79) $(x-4)\, y^4\, dx - x^3(y^2 - 3)\, dy = 0$

80) $x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0, \quad y(1) = \pi/2$

81) $(x^2 + 3y)\, dy + (y + 3x)dy = 0$

82) $y' - xy - x^2 = 0$

83) $y' = -\frac{e^y}{xe^y + 2y}$

84) $(x + y^2)\, y' + y = 0$

85) $y' + \frac{2x\sin y + y^3 e^x}{x^2 \cos y + 3y^2 e^x} = 0$

86) $y' - y = 2xy^{3/2}$

87) $(x+y)\, y' + (y+3x) = 0$

88) $3x(xy-2)dx + (x^3+2y)\, dy = 0$

89) $(3x^2+6xy^2) + (6\, yx^2 + 4y^2)\, y' = 0$

90) $(x+y^2)dy+(y-x^2)\, dx = 0$

91) $(3x^2+4xy)\, dx + (2x^2 + 2y)\, dy = 0$

92) $y' = \frac{2 + ye^{xy}}{2y - xe^{xy}}$

93) $y' = \frac{2xy - y^2}{x^2}$

94) $(x^2 + y) + (e^y + x)\, y' = 0$

95) $(2x^3 - xy^2 - 2y + 3)\,dx - (x^2y + 2x)\,dy = 0$

96) $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

97) $\left[x + \dfrac{1}{\sqrt{y^2 - x^2}}\right] dx + \left[1 - \dfrac{x}{y\sqrt{y^2 - x^2}}\right] dy = 0$

98) $y' = -\dfrac{2x\cos y + 3x^2y}{x^3 - x^2\sin y - y}$

99) $(y + \sqrt{y^2 + x^2})\,dx - x\,dy = 0$

100) $y` = \dfrac{2xye^{(x/y)^2}}{y^2 + y^2e^{(x/y)^2} + 2x^2e^{(x/y)^2}}$

101) $y' = \dfrac{y^3 - 2x^3}{xy^2}$

102) $y' = \dfrac{2y^4 + x^4}{xy^3}$

103) $(x^2 - 3y^2)\,dx + (2xy)\,dy = 0$

104) $\sqrt{y^2 + x^2}\,dx = x\,dy - y\,dx$

105) $(y - xy^2)\,dx + x\,dy = 0$

106) $(y^2 + 2xy)\,dx - x^2\,dy = 0$

107) $y' = \frac{y+1}{x+2y}$

108) $y' = \frac{2x + y\cos^2(y/x)}{x\cos^2(y/x)}$

109) $y' = \frac{x + y\sin^2(y/x)}{x\sin^2(y/x)}$

110) $y` = 3x\, e^{(x+2y)/y}$

111) $(9x^2+2y^2 + 2)\, dx + (4xy + 12\, y^2)\, dy = 0$

112) $(\cos x)\, y` + y = \sin x$

113) $(x+1)\, y` + y = 2x\,(x+1)$

114) $x^2\, y`+ x\, y = x^2 \sin x$

II) Prove that every separable differential equation is exact (Hint: $y` = g(x)\, f(y)$)

III) An RL circuit has an emf of 5 V, a resistance of 50 Ω, an inductance of 1 H, and no initial current. Find the current in the circuit at any time t. Distinguish between the transient and steady-state current.

Answer

The formula is: $R\,i + L\,\frac{di}{dt} = v$, where R is the resistance and L is the inductance, i is the current passes through the circuit, v is voltage source.

After substituting: $50i + 1\,\frac{di}{dt} = 5$, solve this D.E., so that i(t) is called transient current and i(0) is called steady-state current.

IV) A series RL circuit with R = 50 Ω and L = 10 H has a constant voltage V = 100 V applied at t = 0 by the closing of a switch. Find

(a) The equation for i (you may use the formula rather than DE),

(b) The current at t = 0.5 s

(c) The expressions for V_R and V_L

(d) The time at which $V_R = V_L$

V) Solve (x+y) dx + xdy = 0 by two different methods.

VI) Choose the correct answer

A) The general solution of the first order differential equation $t^2y` - 2ty = 3$ is

(a) $\frac{3}{2} + \frac{C}{t^2}$.

(b) $-\frac{3}{2} + \frac{C}{t^2}$.

(c) $Ct^2 - \frac{1}{t}$.

(d) $Ct^2 + \frac{1}{t}$.

B) For the differential equation y` + 2(cosy) y = 1

Which of the following is true?

(a) The differential equation is first order linear and nonhomogeneous.

(b) The differential equation is second order nonlinear and nonhomogeneous.

(c) The differential equation is second order nonlinear and homogeneous.

(d) The differential equation is first order nonlinear and nonhomogeneous.

C) For the differential equation $y' = -\frac{x+y+2}{x-y}$

Which of the following is true?

a) The differential equation is non linear and homogeneous

b) The differential equation is linear and non homogeneous

c) The differential equation is non linear and exact

d) The differential equation is non homogeneous and not exact

D) For the differential equation $y' = -\frac{y-2x(x+1)}{x+1}$

Which of the following is true?

a) The differential equation is non linear and homogeneous

b) The differential equation is linear and non homogeneous

c) The differential equation is linear and non exact

d) The differential equation is non linear and exact

E) For the differential equation $y' = -\frac{9x^2 + 2y^2}{4xy + 12y^2}$

Which of the following is true?

a) The differential equation is linear and homogeneous

b) The differential equation is non linear and non homogeneous

c) The differential equation is linear and exact

d) The differential equation is non linear and exact

F) For the differential equation $y' = -\frac{xy^2 + x}{x^2y + y}$

Which of the following is true?

a) The differential equation is linear and homogeneous

b) The differential equation is linear and non homogeneous

c) The differential equation is separable and non linear

d) The differential equation is non linear and exact

5.4 Homogeneous Higher Order Linear D.E.

As with 2nd order differential equations we can't solve a non homogeneous differential equation unless we can first solve the homogeneous differential equation. We'll also need to restrict ourselves down to constant coefficient differential equations as solving non-constant coefficient differential equations is quite difficult and so we won't be discussing them here. Likewise, we'll only be looking at linear differential equations. So, let's start off with the following differential equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \qquad (12)$$

Now, assume that solutions to this differential equation will be in the form $y = e^{rt}$ and plug this into the differential equation and with a little simplification we get,

$$e^{rt}(a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0) = 0 \qquad (13)$$

and so in order for this to be zero we'll need to require that

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \qquad (14)$$

This is called the characteristic polynomial equation and its roots solutions will give us the solutions to the differential equation. We know that, including repeated roots, an n^{th} degree polynomial (which we have here) will have n roots. So, we need to go through all the possibilities that we've got for roots here.

This is where we start to see differences in how we deal with n^{th} order differential equations versus 2^{nd} order differential equations. There are still the three main cases: **real distinct** roots, **repeated** roots and **complex** roots (although these can now also be repeated as well see). In 2^{nd} order differential equations each differential equation could only involve one of these cases. Now, however, that will not necessarily be the case. We could very easily have differential equations that contain each of these cases.

For instance suppose that we have an 9^{th} order differential equation. The complete list of roots could have 3 roots which only occur once in the list (i.e. real distinct roots), a root with multiplicity 4 (i.e. occurs 4 times in the list) and a set of complex conjugate roots (recall that because the coefficients are all real complex roots will always occur in conjugate pairs).So, for each n^{th} order differential equation we'll need to form a set of n linearly independent functions (i.e. a fundamental set of solutions) in order to get a general solution. In the work that follows we'll discuss the solutions that we get from each case but we will leave it to you to verify that when we put everything together to form a general solution that we do indeed get a fundamental set of solutions. Recall that in order to this we need to verify that the Wronskian is not zero.

So, let's get started with the work here. Let's start off by assuming that in the list of **real distinct** roots of the characteristic equation we have r_1, r_2, …, r_k and they only occur once in the list. The solution from each of these will then be, $e^{r_1 x}, e^{r_2 x}, ..., e^{r_k x}$.

Now let's take a look at **repeated** roots. The result here is a natural extension of the work we saw in the 2^{nd} order case. Let's suppose that r is a root of multiplicity k (i.e. r occurs k times in the list of roots). We will then get the following k solutions to the differential equation, $e^{rx}, xe^{rx}, ..., x^{k-1}e^{rx}$. So, for repeated roots we just add in a x for each of the solutions past the first one until we have a total of k solutions. Again, we will leave it to you to compute the Wronskian to verify that these are in fact a set of linearly independent solutions. Finally we need to deal with **complex** roots; the biggest issue here is that we can now have repeated complex roots for 4th order or higher differential equations. We'll start off by assuming that $r = \lambda + \mu i$ occurs only once in the list of roots. In this case we'll get the standard two solutions, $e^{\lambda x}\cos\mu x, e^{\lambda x}\sin\mu x$. Now let's suppose that $r = \lambda + \mu i$ has a multiplicity of k (i.e. they occur k times in the list of roots). In this case we can use the work from the repeated roots above to get the following set of 2k complex-valued solutions, $e^{(\lambda+\mu i)x}, xe^{(\lambda+\mu i)x}, ... \; x^{k-1}e^{(\lambda+\mu i)x}$. The problem here of course is that we really want real-valued solutions. So, recall that in the case where they occurred once all we had to do was use Euler's formula on the first one and then take the real and imaginary part to get two real valued solutions. We'll do the same thing here and use Euler's formula on the first set of complex-valued solutions above, split each one into its real and imaginary parts to arrive at the following set of 2k real-valued solutions $e^{\lambda x}\cos\mu x, e^{\lambda x}\sin\mu x,\ xe^{\lambda x}\cos\mu x,\ xe^{\lambda x}\sin\mu x, ..., x^{k-1}e^{\lambda x}\cos\mu x,$

$x^{k-1}e^{\lambda x}\sin\mu x$. Once again we'll leave it to you to verify that these do in fact form a fundamental set of solutions. Before we work a couple of quick examples here we should point out that the characteristic polynomial is now going to be at least a 3rd degree polynomial and finding the roots of these by hand is often a very difficult and time consuming process and in many cases if the roots are not rational (i.e. in the form $\frac{p}{q}$) it can be almost impossible to find them all by hand. To see a process for determining all the rational roots of a polynomial check out the finding zeros of polynomials page in my Algebra notes. In practice however, we usually use some form of computation aid such as Maple or Mathematica to find all the roots. So, let's work a couple of example here to illustrate at least some of the ideas discussed here.

Example 13: Solve the following I.V.P.

$$y''' - 5y'' - 22y' + 56y = 0, y(0) = 1, y'(0) = -2, y''(0) = -4$$

Solution: The characteristic equation is,

r^3- 5 r^2- 22r +56 = (r+4) (r-2) (r-7), from which r = -4, r = 2, r = 7, so we have three real distinct roots hear and so the general solution is $y(t) = c_1e^{-4x} + c_2e^{2x} + c_3e^{7x}$

Differentiating a couple of times and applying the initial conditions give the following system of equations that we'll need to solve in order to find the coefficients.

1= y(0) = $c_1 + c_2 + c_3$, -2 = $y'(0) = -4c_1 + 2c_2 + 7c_3$, - 4 = $y''(0) = 16c_1 + 4c_2 + 49c_3$, thus $c_1 = \frac{14}{33}$, $c_2 = \frac{13}{15}$, $c_3 = \frac{-16}{55}$. The actual solution is then, $y(t) = \frac{14}{33}e^{-4x} + \frac{13}{15}e^{2x} + \frac{-16}{55}e^{7x}$.

Example 14: Solve the following differential equation.

$$2y'''' + 11y''' + 18y'' + 4y' - 8y = 0$$

Solution: The characteristic equation is

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0$$

And so we have two roots here, $r_1 = 1/2$ and $r_2 = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first we add x's only the solution until we reach three solutions, then the general solution is,

$$y(t) = c_1 e^{(1/2)x} + e^{-2x}[c_2 + c_3 x + c_4 x^2]$$

Example 15: Solve the following differential equation.

$$y^{(5)} + 12y'''' + 104y''' + 408y'' + 1156y' = 0$$

Solution: The characteristic equation is

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r(r^2 + 6r + 34)^2 = 0$$

So, we have one real root r = 0 and a pair of complex roots r = -3 $\pm$ 5i each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the normal solutions and two will be the normal solution each multiplied by x. The general solution is:

$y(t) = c_1 + e^{-3x} [c_2 \cos(5x) + c_3 \sin(5x)] + xe^{-3x} [c_4 \cos(5x) + c_5 \sin(5x)]$

Example 16: Solve the following differential equation.

$$y^{(5)} - 15y'''' + 84y''' - 220y'' + 275y' - 125y = 0$$

Solution: The characteristic equation is, $r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2(r^2 - 4r + 5) = 0$

In this case we've got one real distinct root r = 1 and double root r = 5 and a pair of complex roots, r = 2 $\pm$ i that only occur once, hence the general solution is then,

$$y(x) = c_1 e^t + e^{5x} [c_2 + c_3 x] + e^{2x} [c_4 \cos(x) + c_5 \sin(x)]$$

Example 17: Solve the following differential equation

$$y'''' + 16y = 0$$

Solution: The characteristic equation is $r^4 + 16 = 0$, from which $r = 2\,e^{i(\pi+2\pi k)/4}$, k=0,1,2,3, thus the roots are $r = 2e^{i\pi/4}$, $2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}$, thus $r = \pm\sqrt{2} + i\sqrt{2}$, $r = \pm\sqrt{2} - i\sqrt{2}$. The general solution is then, y(t) =

$$e^{\sqrt{2}x}(c_1 \cos(\sqrt{2})x + c_2 \sin(\sqrt{2})x) + e^{(-\sqrt{2})x}(c_1 \cos(\sqrt{2})x + c_2 \sin(\sqrt{2})x)$$

5.5 Non Homogeneous Linear second Order D.E.

It's now time to start thinking about how to solve non homogeneous differential equations. A second order, linear non homogeneous differential equation is

$$y'' + p(x)\,y' + q(x)y = g(x) \qquad (15)$$

$g(x)$ is a non-zero function. Note that we didn't go with constant coefficients here because everything that we're going to do in this section doesn't require it. Also, we're using a coefficient of 1 on the second derivative just to make some of the work a little easier to write down. It is not required to be a 1. Before talking about how to solve one of these we need to get some basics out of the way, which is the point of this section. First, we will call

$$y'' + p(x)y' + q(x)y = 0 \qquad (16)$$

(16) is the associated homogeneous differential equation to (15). Now, let's take a look at the following theorem.

Theorem 1

Suppose that $Y_1(x)$ and $Y_2(x)$ are two solutions to (15) and that $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions to the associated homogeneous differential equation (16) then, $Y_1(x) - Y_2(x)$ is a solution to (16) and it can be written as:

$$Y_1(x) - Y_2(x) = c_1\, y_1(x) + c_2\, y_2(x) \qquad (17)$$

Proof

Note the notation used here. Capital letters referred to solutions to (15) while lower case letters referred to solutions to (16). This is a fairly common convention when dealing with non homogeneous differential equations. This theorem is easy enough to prove so let's do that. To prove that $Y_1(x)$ - $Y_2(x)$ is a solution to (16) all we need to do is plug this into the differential equation and check it. We used the fact that $Y_1(x)$ and $Y_2(x)$ are two solutions to (15) such that $Y_1'' + p(x)\,Y_1' + q(x)Y_1 = g(x), Y_2'' + p(x)\,Y_2' + q(x)Y_2 = g(x)$, therefore $(Y_1 - Y_2)'' + p(x)\,(Y_1 - Y_2)' + q(x)\,(Y_1 - Y_2)$

$$= Y_1'' + p(x)\,Y_1' + q(x)\,Y_1 - (Y_2'' + p(x)\,Y_2' + q(x)Y_2)$$

$$= g(x) - g(x) = 0$$

So, we were able to prove that the difference of the two solutions is a solution to (16).

Proving that $Y_1(x) - Y_2(x) = c_1\, y_1(x) + c_2\, y_2(x)$ is even easier.

Since $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (16) we know that they form a general solution and so any solution to (16) can be written in the form

$$y(t) = c_1\, y_1(x) + c_2\, y_2(x).$$

Well, $Y_1(x) - Y_2(x)$ is a solution to (16), as we've shown above, therefore it can be written as $Y_1(x) - Y_2(x) = c_1\, y_1(x) + c_2\, y_2(x)$, So, what does this theorem do for us? We can use this theorem to write down the form of the general solution to (15). Let's suppose that $y(t)$ is the general solution to (15) and that $Y_P(x)$is any solution to (15) that we can get our hands on. Then using the second part of our theorem we know that $y(x) - Y_p(x) = c_1\, y_1(x) + c_2\, y_2(x)$, where $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions for (5). Solving for $y(x)$ gives,

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) + Y_p(x) \qquad (18)$$

We will call

$$y_c(t) = c_1\, y_1(x) + c_2\, y_2(x) \qquad (19)$$

is the complimentary solution and $Y_P(t)$ a particular solution. The general solution to a differential equation can then be written as.

$$y(x) = y_c(x) + Y_p(x) \qquad (20)$$

So, to solve a non homogeneous differential equation, we will need to solve the homogeneous differential equation, (16), which for constant coefficient differential equations is pretty easy to do, and we'll need a solution to (15). This seems to be a circular argument. In order to write down a solution to (15) we need a solution. However, this isn't the problem that it seems to be. There are ways to find a solution to (15). They just won't, in general, be the general solution. In fact, the next two sections are devoted to exactly that, finding a particular solution to a non homogeneous differential equation. There are two common methods for finding particular solutions: Undetermined Coefficients and Variation of Parameters. Both have their advantages and disadvantages. g(x) may be e^{ax}, cosax, sinax, x^n, x^n cos(ax), x^n sin(ax), e^{ax} cos(ax), e^{ax} sin(ax), therefore we will study the particular solution for all these functions using the D - operator method.

In general the non homogeneous higher order differential equation will be in the form

$$A_n y^{(n)} + A_{n-1}\, y^{(n-1)} + A_{n\text{-}2}\, y^{(n-2)} + \ldots + A_1\, y' + A_0 y = g(x) \qquad (21)$$

This equation has another form as follows

$$(A_n D^n + A_{n-1}\, D^{n-1} + A_{n\text{-}2}\, D^{n-2} + \ldots + A_1\, D + A_0)y = g(x) \qquad (22)$$

5.6 D - Operator

$D = \dfrac{d}{dx}$ is called differential operator satisfying the following properties:

- $D(a\, f(x) + b\, g(x)) = a\, D\, f(x) + b\, Dg(x)$;
- $D^n f(x) = f^{(n)}(x)$;
- $(D^n + D^m) f(x) = f^{(n)}(x) + f^{(m)}(x)$;
- $\dfrac{1}{D}$ is called inverse operator of D where $\dfrac{1}{D} f(x) = \int f(x)\, dx$;
- $\dfrac{1}{\phi(D)} e^{ax} = \dfrac{1}{\phi(a)} e^{ax}, \phi(a) \neq 0$, $\phi(D)$ is a polynomial of differential operator D;
- $\dfrac{1}{\phi(D^2)} \cos(ax) = \dfrac{1}{\phi(-a^2)} \cos(ax), \phi(-a^2) \neq 0$;
- $\dfrac{1}{\phi(D^2)} \sin(ax) = \dfrac{1}{\phi(-a^2)} \sin(ax), \phi(-a^2) \neq 0$;
- $\dfrac{1}{\phi(D^2)} e^{ax} \cos(ax) = e^{ax} \dfrac{1}{\phi[(D+a)^2]} \cos(ax)$;

- $\frac{1}{\phi(D^2)} e^{ax}\sin(ax) = e^{ax}\frac{1}{\phi[(D+a)^2]}\sin(ax)$

Example 18: Evaluate

i) $D(3x^2)$, $D^3(x^4)$, $\frac{1}{D}x$, $(D+5)^2(6x^3)$, ii) $\frac{1}{D^2+5D-3}e^{2x}$,

iii) $\frac{1}{D^2+3}\cos(4x)$, iv) $\frac{1}{D^2+7}\sin(5x)$

Solution:

i) $D(3x^2) = 6x$, $D^3(x^4) = 24x$, $\frac{1}{D}x = \int x\,dx = \frac{x^2}{2}$,

$(D+5)^2(6x^3) = (D^2+10D+25)(6x^3) = 36x+180x^2+150x^3$

ii) $\frac{1}{D^2+5D-3}e^{2x} = \frac{1}{2^2+5(2)-3}e^{2x} = \frac{e^{2x}}{11}$

iii) $\frac{1}{D^2+3}\cos(4x) = \frac{1}{-4^2+3}\cos(4x) = -\frac{\cos(4x)}{13}$

iv) $\frac{1}{D^2+7}\sin(5x) = \frac{1}{-5^2+7}\sin(5x) = -\frac{\sin(5x)}{18}$

The solution of equation (22) has 2 parts homogeneous and particular parts.

5.7 Particular integral

We will study the particular solution of n^{th} order differential equation with constant coefficient (equation 22) if

a) $g(x) = e^{ax}$ such that

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + A_{n-2} y^{(n-2)} + ... + A_1 y' + A_0 y = e^{ax} \qquad (23)$$

It can be written in the form

$$(A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + ... + A_1 D + A_0)y = e^{ax} \qquad (24)$$

We have to get first the homogeneous part of the solution $y_c(t)$ so that

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + A_{n-2} y^{(n-2)} + ... + A_1 y' + A_0 y = 0$$

The characteristic equation is

$$A_n r^n + A_{n-1} r^{n-1} + A_{n-2} r^{n-2} + ... + A_1 r + A_0 = 0$$

So that we can get the roots of the equation from which we get the homogeneous part of the solution.

To get the particular solution, we have to follow these steps

(1) Replace $y^{(n)}$ by $D^n y$, in equation (23)

(2) We will get

$$(A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + ... + A_1 D + A_0)y = e^{ax}$$

(3) Hence the particular solution is expressed as

$$Yp(x) = \frac{1}{A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + ... + A_1 D + A_0} e^{ax}$$

but $\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}$, hence

$$Y_p(x) = \frac{1}{A_n (a)^n + A_{n-1} (a)^{n-1} + A_{n-2} (a)^{n-2} + ... + A_1 (a) + A_0} e^{ax}$$

(4) Since the solution is $y(x) = y_c(x) + Y_p(x)$

If $A_n (a)^n + A_{n-1} (a)^{n-1} + A_{n-2} (a)^{n-2} + ... + A_1 (a) + A_0 = 0$, then apply the following theorem.

Theorem 2

Let $\phi(D)$ be a polynomial of differential operator D and f(x) be differentiable function, then

$$\frac{1}{\phi(D)}e^{ax}f(x)=e^{ax}\frac{1}{\phi(D+a)}f(x)$$

Proof

Let $\frac{1}{\phi(D+a)}f(x) = g(x)$, i.e. $f(x) = \phi(D+a)\,g(x)$, but $\phi(D)\,e^{ax}$ $g(x) = e^{ax}\,\phi(D+a)\,g(x) = e^{ax}f(x)$, thus $e^{ax}\,g(x) = \frac{1}{\phi(D)}e^{ax}f(x)$, hence $e^{ax}\frac{1}{\phi(D+a)}f(x) = \frac{1}{\phi(D)}e^{ax}f(x)$.

Example 19: Find the particular solution for the differential equation y``-8y`+16y= xe^{4x}

Solution

$$\frac{1}{(D-4)^2}xe^{4x}=e^{4x}\frac{1}{(D+4-4)^2}x=e^{4x}\frac{1}{D^2}x=e^{4x}(\frac{x^3}{6})$$

Example 20: Solve the differential equation

$$y'' - 3y' - 4y = e^{5x}$$

Solution: The characteristic equation is $r^2 - 3r - 4 = 0 \Rightarrow$ $(r-4)(r+1) = 0 \Rightarrow r = 4, -1$,therefore $y_c = c_1 e^{4x} + c_2 e^{-x}$ and

$$Y_p(x) = \frac{1}{D^2 - 3D - 4} e^{5x} = \frac{1}{5^2 - 3(5) - 4} e^{5x} = \frac{e^{5x}}{6}.$$

Hence the solution is $y(x) = y_c + y_p = c_1 e^{4x} + c_2 e^{-x} + \dfrac{e^{5x}}{6}$

For the D.E. $\phi(D)y = e^{ax}$, $y_p = \dfrac{1}{\phi(D)} e^{ax} = \dfrac{1}{\phi(a)} e^{ax}$.

If $\phi(a) = 0$, thus we treat this problem such that

$$y_p = \frac{1}{\phi(D)} e^{ax} = e^{ax} \frac{1}{\phi(D + a)}(1).$$

This case will be discussed in the following examples

Example 21: Solve the differential equation

$$y'' - 4y' + 4y = e^{2x}$$

Solution: It is so easy to get $y_c = c_1e^{2x} + c_2xe^{2x}$ as the roots of characteristic equation $r^2 - 4r + 4 = 0$ are r = -2, -2 and

$$Y_p(x) = \frac{1}{D^2 - 4D + 4} e^{2x} = \frac{1}{(D-2)^2} e^{2x} = e^{2x} \frac{1}{(D+2-2)^2} e^{0x}$$

$$= e^{2x} \frac{1}{D^2}(1) = \frac{x^2}{2} e^{2x}$$

Example 22: Solve the differential equation

$$y'' - 4y' + 3y = e^{3x}$$

Solution: $y_c = c_1e^{3x} + c_2e^{x}$ & $Y_p(x) = \frac{1}{(D-3)(D-1)} e^{3x}$

$$= \frac{1}{2(D-3)} e^{3x} = \frac{e^{3x}}{2} \left(\frac{1}{(D+3-3)}\right) e^{0x} = \frac{e^{3x}}{2} \frac{1}{D}(1) = \frac{x\,e^{3x}}{2}.$$

b) g(x) = cos(ax) such that

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + A_{n-2} y^{(n-2)} + \ldots + A_1 y' + A_0 y = \cos(ax) \quad (25)$$

It can be written in the form

$$(A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + ... + A_1 D + A_0)y = \cos(ax) \quad (26)$$

After we get the homogeneous part of (26), such that the characteristic equation is

$$A_n r^n + A_{n-1} r^{n-1} + A_{n-2} r^{n-2} + ... + A_1 r + A_0 = 0$$

To get the particular solution, we have to put $\phi(D)$ as function of D^2 such that if n is even in (26), and then the equation is expressed as $(A_n (D^2)^{n/2} + A_{n-1} D(D^2)^{(n-2)/2} + A_{n-2} (D^2)^{(n-2)/2}$

$+ ... + A_2 D^2 + A_1 D + A_0)y = \cos(ax)$

Therefore $Y_p(x) =$

$$\frac{1}{A_n(-a^2)^{n/2} + A_{n-1}D(-a^2)^{(n-2)/2} + A_{n-2}(-a^2)^{(n-2)/2} + ... + A_1 D + A_0} \cos(ax)$$

If n is odd in (26), then the equation is expressed as

$$(A_n D(D^2)^{(n-1)/2} + A_{n-1}(D^2)^{(n-1)/2} + A_{n-2} D(D^2)^{(n-3)/2} + ... + A_1 D$$
$$+ A_0) y = \cos(ax)$$

Thus $Y_p(x) =$

$$\frac{1}{A_n D(-a^2)^{(n-1)/2} + A_{n-1}(-a^2)^{(n-1)/2} + ... + A_1 D + A_0} \cos(ax)$$

Similarly For **g(x) = sin(ax)**

If n is even, then $Y_p(x) =$

$$\frac{1}{A_n(-a^2)^{n/2} + A_{n-1} D(-a^2)^{(n-2)/2} + ... + A_1 D + A_0} \sin(ax)$$

If n is odd, then $Y_p(x) =$

$$\frac{1}{A_n D(-a^2)^{(n-1)/2} + A_{n-1}(-a^2)^{(n-1)/2} + ... + A_1 D + A_0} \sin(ax)$$

Note

For $Y_p(x) = \frac{1}{A_n D^n + A_{n-1} D^{n-1} + ... + A_1 D + A_0} \cos(ax)$

If n is even and

$$A_n(-a^2)^{n/2} + A_{n-1}D(-a^2)^{(n-2)/2} + ... + A_1D + A_0 = 0,$$

Or if n is odd and

$$A_nD(-a^2)^{(n-1)/2} + A_{n-1}(-a^2)^{(n-1)/2} + ... + A_1D + A_0 = 0$$

Then put cos(ax) = Re (e^{iax}) to evaluate $Y_p(x)$ such that:

$$Y_p(x) = \frac{1}{A_nD^n + A_{n-1}D^{n-1} + ... + A_1D + A_0}\cos(ax)$$

$$= \frac{1}{A_nD^n + A_{n-1}D^{n-1} + ... + A_1D + A_0}\operatorname{Re}(e^{iax})$$

Therefore Yp(x) =

$$\operatorname{Re}(e^{iax})\frac{1}{A_n(D+ia)^n + A_{n-1}(D+ia)^{n-1} + ... + A_1(D+ia) + A_0}e^{0x}$$

Thus Yp(x) =

$$\operatorname{Re}(e^{iax})\frac{1}{A_n(0+ia)^n + A_{n-1}(0+ia)^{n-1} + ... + A_1(0+ia) + A_0}$$

$$= \operatorname{Re}(e^{iax})\frac{1}{A_n(ia)^n + A_{n-1}(ia)^{n-1} + ... + A_1(ia) + A_0}$$

Similarly, if

$$Y_p(x) = \frac{1}{A_nD^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + ... + A_1D + A_0}\sin(ax)$$

If n is even and

$$A_n(-a^2)^{n/2} + A_{n-1}D(-a^2)^{(n-2)/2} + ... + A_1D + A_0 = 0$$

Or if n is odd and

$$A_nD(-a^2)^{(n-1)/2} + A_{n-1}(-a^2)^{(n-1)/2} + ... + A_1D + A_0 = 0$$

Then put $\sin(ax) = \operatorname{Im}(e^{iax})$ to evaluate $Y_p(x)$ such that :

$Y_p(x)$

$$= \frac{1}{A_nD^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + ... + A_1D + A_0}\sin(ax)$$

$$= \frac{1}{A_nD^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + ... + A_1D + A_0}\operatorname{Im}(e^{iax})$$

Therefore Yp(x)

$$\text{Im}(e^{iax})\frac{1}{A_n(D+ia)^n + A_{n-1}(D+ia)^{n-1} + \ldots + A_1(D+ia) + A_0} e^{0x}$$

Thus Yp(x)

$$= \text{Im}(e^{iax})\frac{1}{A_n(0+ia)^n + A_{n-1}(0+ia)^{n-1} + \ldots + A_1(0+ia) + A_0}$$

$$= \text{Im}(e^{iax})\frac{1}{A_n(ia)^n + A_{n-1}(ia)^{n-1} + \ldots + A_1(ia) + A_0}$$

Example 23: Solve the D.E. $y'' + 4y = \cos 4x$

Solution: It is so easy to get $y_c = c_1 \cos 2x + c_2 \sin 2x$, since the roots of characteristic equation $r^2 + 4 = 0$ are $r = \pm 2i$ and $Y_p(x) = \frac{1}{D^2+4}\cos 4x = \frac{1}{-16+4}\cos 4x = \frac{1}{-12}\cos 4x$.

Example 24: Solve the D.E. $y'' + 4y' + 8y = \cos 2x$

Solution: Since the characteristic equation is $r^2+4r+8=0$, then the roots are $r = -2 \pm 2i$, thus $y_c = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x)$

and $Y_p(x) = \dfrac{1}{D^2+4D+8}\cos 4x = \dfrac{1}{-16+4D+8}\cos 4x$

$$= \frac{1}{4(D-2)}\cos 4x = \frac{(D+2)}{4(D^2-4)}\cos 4x = \frac{-4\sin 4x + 2\cos 4x}{4(-16-4)}$$

$$= \frac{2\sin 4x - \cos 4x}{40}$$

c) g(x) = x^m such that

$$A_n y^{(n)} + A_{n-1}\, y^{(n-1)} + A_{n-2}\, y^{(n-2)} + ... + A_1\, y' + A_0 y = x^m \qquad (27)$$

It can be written in the form

$$(A_n D^n + A_{n-1}\, D^{n-1} + A_{n-2}\, D^{n-2} + ... + A_1\, D + A_0)y = x^m \qquad (28)$$

From eq. 28 , we can get the particular solution, as follows

$$Y_p(x) = \frac{1}{A_0}\left[\frac{1}{1+\dfrac{A_n D^n + A_{n-1}\, D^{n-1} + A_{n-2}\, D^{n-2} + ... + A_1\, D}{A_0}}\right] x^m,$$

$$= \frac{1}{A_0}\left[1+\frac{A_n D^n + A_{n-1}\, D^{n-1} + A_{n-2}\, D^{n-2} + ... + A_1\, D}{A_0}\right]^{-1} x^m$$

Note: The expansion depends on **m**, i.e. if m=1, then

$$[1+\frac{A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + ... + A_1 D}{A_0}]^{-1} = [1-\frac{A_1 D}{A_0}]$$

Example 25: Solve the D.E. $y'' + 4y' + 3y = x^2$

Solution: Since the characteristic equation is $r^2+4r+3=0$, then the roots are $r = 1, 3$, thus $y_c = c_1 e^x + c_2 e^{3x}$ and $Y_p(x) =$

$$\frac{1}{D^2+4D+3}x^2 = \frac{1}{3}(\frac{1}{1+\frac{D^2+4D}{3}})x^2 = \frac{1}{3}(1+\frac{D^2+4D}{3})^{-1}x^2$$

$$= \frac{1}{3}(1-\left[\frac{D^2+4D}{3}\right]+\left[\frac{D^2+4D}{3}\right]^2+...)x^2$$

$$= \frac{1}{3}(1-\left[\frac{D^2+4D}{3}\right]+\left[\frac{16D^2}{9}\right])x^2 = \frac{1}{3}(x^2-\left[\frac{2+8x}{3}\right]+\left[\frac{32}{9}\right])$$

$$= \frac{1}{3}(x^2 - \frac{8x}{3} + \frac{26}{9})$$

Example 26: Solve the D.E. $y'' + 4y = \cos 2x$

Solution: Since the characteristic equation is $r^2 + 4 = 0$, then the roots are $r = \pm 2i$, thus $y_c = (c_1 \cos 2x + c_2 \sin 2x)$

and $Y_p(x) = \frac{1}{D^2+4}\cos 2x = \frac{1}{D^2+4}\mathrm{Re}(e^{i2x})$

$$= \mathrm{Re}(e^{i2x})\frac{1}{(D+2i)^2+4}e^{0x} = \mathrm{Re}(e^{i2x})\frac{1}{D^2+4iD}e^{0x}$$

$$= \mathrm{Re}(e^{i2x})\frac{1}{(D+4i)D}e^{0x} = \mathrm{Re}(e^{i2x})\frac{1}{(D+4i)}x$$

$$= \frac{\mathrm{Re}(e^{i2x})}{4i}\frac{1}{(1-\frac{iD}{4})}x = \frac{\mathrm{Re}(e^{i2x})}{4i}(1-\frac{iD}{4})^{-1}x$$

$$= \frac{-\mathrm{Re}(ie^{i2x})}{4}(1+\frac{iD}{4})x = \frac{-\mathrm{Re}(ie^{i2x})}{4}(x+\frac{i}{4}) = \frac{-\mathrm{Re}(e^{i2x})}{4}(ix-\frac{1}{4})$$

$$= \frac{-\mathrm{Re}(\cos 2x + i\sin 2x)}{4}(ix-\frac{1}{4}) = \frac{x\sin 2x}{4} + \frac{\cos 2x}{16}$$

d) $g(x) = e^{ax}\cos bx$ such that

$$A_n y^{(n)} + A_{n-1}y^{(n-1)} + A_{n-2}y^{(n-2)} + \ldots + A_1 y' + A_0 y = e^{ax}\cos bx \quad (29)$$

It can be written in the form

$$(A_n D^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + \ldots + A_1 D + A_0)y = e^{ax}\cos bx \quad (30)$$

From eq. (30) and according to theorem 2, we can get the particular solution, as follows

$Y_p(x)$

$$= \frac{1}{A_n D^n + A_{n-1} D^{n-1} + A_{n-2} D^{n-2} + \ldots + A_1 D + A_0} e^{ax}\cos(bx)$$

$$= e^{ax} \frac{1}{A_n (D+a)^n + A_{n-1} (D+a)^{n-1} + \ldots + A_1 (D+a) + A_0} \cos(bx)$$

Example 27: Solve the D.E. $y'' + 5y' + 6y = e^{3x}\cos 2x$

Solution: The characteristic equation is $r^2 + 5r + 6 = 0 \Rightarrow$ (r+2)(r+3) = 0 $\Rightarrow$ r = -2, -3, thus $y_c = c_1 e^{-2x} + c_2 e^{-3x}$ & $Y_p(x) =$

$$\frac{1}{D^2 + 4D + 6} e^{3x} \cos 2x = e^{3x} \frac{1}{(D+3)^2 + 4(D+3) + 6} \cos 2x$$

$$= e^{3x} \frac{1}{D^2 + 10D + 27} \cos 2x = e^{3x} \frac{1}{(-2^2) + 10D + 27} \cos 2x$$

$$= e^{3x} \frac{1}{10D + 23} \cos 2x = e^{3x} \frac{(10D - 23)}{100D^2 - 529} \cos 2x$$

$$= e^{3x} \frac{(-20\sin 2x - 23\cos 2x)}{100(-2^2) - 529} = e^{3x} \frac{(20\sin 2x + 23\cos 2x)}{-929}$$

The same steps are followed if $\mathbf{g(x) = e^{ax} \sin bx}$ except cosbx is replaced by sinbx.

Example 28: Solve the D.E. $y'' + 9y = e^{-x}\sin 3x$

Solution: The characteristic equation is $r^2 + 9 = 0 \Rightarrow r = \pm 3i$, thus $y_c = (c_1 \cos 3x + c_2 \sin 3x)$ and

$$Yp(x) = \frac{1}{D^2+9} e^{-x} \sin 3x = e^{-x} \frac{1}{(D-1)^2 + 4(D-1) + 9} \sin 3x$$

$$= e^{-x} \frac{1}{D^2 + 2D + 6} \sin 3x = e^{-x} \frac{1}{D^2 + 2D + 6} \sin 3x$$

$$= e^{-x} \frac{1}{-3^2 + 2D + 6} \sin 3x = e^{-x} \frac{1}{2D-3} \sin 3x = e^{-x} (\frac{2D+3}{4D^2 - 9}) \sin 3x$$

$$= e^{-x} (\frac{6\cos 3x + 3\sin 3x}{4(-3^2) - 9}) = -e^{-x} (\frac{2\cos 3x + \sin 3x}{15})$$

e) $\mathbf{g(x) = e^{ax} x^m}$ such that

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + A_{n-2} y^{(n-2)} + ... + A_1 y' + A_0 y = e^{ax} x^m \qquad (31)$$

It can be written in the form

$$(A_nD^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + ... + A_1D + A_0)y = e^{ax}x^m \qquad (32)$$

From eq. (32) , we can get the particular solution, as follows

$$Y_p(x) = \frac{1}{A_nD^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + ... + A_1D + A_0} e^{ax}x^m$$

$$= e^{ax}\frac{1}{A_n(D+a)^n + A_{n-1}(D+a)^{n-1} + ... + A_1(D+a) + A_0} x^m$$

$$= \frac{e^{ax}}{A_0}\left(\frac{1}{1+\frac{A_n(D+a)^n + A_{n-1}(D+a)^{n-1} + ... + A_1(D+a)}{A_0}}\right)x^m$$

$$= \frac{e^{ax}}{A_0}\left(1+\frac{A_n(D+a)^n + A_{n-1}(D+a)^{n-1} + ... + A_1(D+a)}{A_0}\right)^{-1} x^m$$

Example 29: Solve the D.E. $y'' + 6y' + 9y = e^{3x}x^2$

Solution: The characteristic equation is $r^2 + 6r + 9 = 0 \Rightarrow$ $r = -3, -3$, thus $y_c = c_1 e^{-3x} + c_2 xe^{-3x}$,

$$Y_p(x) = \frac{1}{D^2+6D+9} e^{3x} x^2 = e^{3x} \frac{1}{(D+3)^2+6(D+3)+9} x^2$$

$$= e^{3x} \frac{1}{D^2+12D+36} x^2 = e^{3x} \frac{1}{(D+6)^2} x^2 = \frac{e^{3x}}{36}(1+\frac{D}{6})^{-2} x^2$$

$$= \frac{e^{3x}}{36}(1-\frac{2D}{6}+\frac{3D^2}{36}-...)x^2 = \frac{e^{3x}}{36}(x^2-\frac{2x}{3}+\frac{1}{6}).$$

5.8 Variation of Parameter

The method of Variation of Parameters is a much more general method that can be used in many more cases. However, there are two disadvantages to the method. First, the complimentary solution is absolutely required to do the problem. This is in contrast to the method of undetermined coefficients where it was advisable to have the complimentary solution on hand, but was not required. Second, as we will see, in order to complete the method we will be doing a couple of integrals and there is no guarantee that we will be able to do the integrals. So, while it will always be possible to write down a formula to get the particular solution, we may not be able to actually find it if the integrals are too difficult or if we are unable to find the complimentary solution. We're going to derive the formula for variation of parameters. Assume non homogeneous 2nd order D.E. is

$$p(x)y'' + q(x)\,y' + r(x)y = g(x) \qquad (33)$$

Let its complimentary solution is

$$y_c(x) = c_1\,y_1(x) + c_2\,y_2(x)$$

Remember as well that this is the general solution to the homogeneous differential equation.

$$p(x)y'' + q(x)\,y' + r(x)y = 0 \qquad (34)$$

Also recall that in order to write down the complimentary solution we know that $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions. What we're going to do is see if we can find a pair of functions, $u_1(x)$ and $u_2(x)$ so that

$$Y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \qquad (35)$$

will be a solution to (33). We have two unknowns here and so we'll need two equations eventually. One equation is easy. Our proposed solution must satisfy the differential equation, so we'll get the first equation by plugging our proposed solution into (33). The second equation can come from a variety of places. We are going to get our second equation simply by making an assumption that will make our work easier. We'll say more about this shortly. So, let's start. If we're going to plug our proposed solution into the differential equation we're going to need some derivatives so let's get those. The first derivative is

$$Y_P'(x) = u_1'(x)y_1(x) + u_1(x)\,y_1'(x) + u_2'(x)y_2(x) + u_2(x)\,y_2'(x)$$

Here's the assumption. Simply to make the first derivative easier to deal with we are going to assume that whatever $u_1(x)$ and $u_2(x)$ are they will satisfy the following.

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \qquad (36)$$

Now, there is no reason ahead of time to believe that this can be done. However, we will see that this will work out. We simply make this assumption on the hope that it won't cause problems down the road and to make the first derivative easier so don't get excited about it. With this assumption the first derivative becomes.

$$Y_P'(x) = u_1(x)\, y_1'(x) + u_2(x)\, y_2'(x) \tag{37}$$

The second derivative is then,

$$Y_P''(x) = u_1'(x)\, y_1'(x) + u_1(x) y_1''(x) + u_2'(x)\, y_2'(x) + u_2(x) y_2''(x)$$

Plug the solution and its derivatives into (33) such that

$$p(x)[u_1'\, y_1' + u_1 y_1'' + u_2'\, y_2' + u_2 y_2''] + q(x)[u_1\, y_1' + u_2\, y_2']$$

$$+ r(x)[u_1 y_1 + u_2 y_2] = g(x)$$

Rearranging a little gives the following.

$$p(x)[u_1' y_1' + u_2' y_2'] + u_1(x)[p(x)y_1'' + q(x)y_1' + r(x)y_1]$$

$$+u_2(x)[p(x)y_2'' + q(x)y_2' + r(x)y_2] = g(x)$$

Now, both $y_1(x)$ and $y_2(x)$ are solutions to (34) and so the second and third terms are zero. Acknowledging this and rearranging a little gives us,

$$p(x)[u_1' y_1' + u_2' y_2'] + u_1(x)[0] + u_2(x)[0] = g(x)$$

Therefore

$$u_1' y_1' + u_2' y_2' = \frac{g(x)}{p(x)} \tag{38}$$

We've almost got the two equations that we need. Before proceeding we're going to go back and make a further assumption. The last equation (38), is actually the one that we want, however, in order to make things simpler for us we are going to assume that the function p(x) = 1.

In other words, we are going to go back and start working with the D.E. $p(x)y'' + q(x)\,y' + r(x)y = g(x)$. If the coefficient of the second derivative isn't one divide it out so that it becomes a one. The formula that we're going to be getting will assume this! Upon doing this the two equations that we want so solve for the unknown functions are

$$u_1' y_1 + u_2' y_2 = 0 \tag{39}$$

$$u_1'\, y_1' + u_2'\, y_2' = g(x) \tag{40}$$

Note that in this system we know the two solutions and so the only two unknowns here are u_1' and u_2'. Solving this system is actually quite simple. First, solve (39) for u_1' and plug this into (40) and do some simplification.

$$u_1' = -\frac{u_2' y_2}{y_1} \tag{41}$$

$$-\frac{u_2' y_2}{y_1} y_1' + u_2' y_2' = g(x) \Rightarrow u_2'(y_2' - \frac{y_2 y_1'}{y_1}) = g(x) \Rightarrow$$

$$u_2'(\frac{y_2' y_1 - y_2 y_1'}{y_1}) = g(x)$$

$$u_2' = \frac{y_1 g(x)}{y_2' y_1 - y_2 y_1'}$$

(42)

So, we now have an expression for u_2' . Plugging this into (28) will give us an expression for u_1' .

$$u_1' = -\frac{y_2 g(x)}{y_2' y_1 - y_2 y_1'} \qquad (43)$$

Next, let's notice that $W(y_1, y_2) = y_2' y_1 - y_2 y_1' \neq 0$

Recall that $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions and so we know that the Wronskian won't be zero!

Finally, all that we need to do is integrate (42) and (43) in order to determine what $u_1(x)$ and $u_2(x)$ are. Doing this gives,

$$u_1(x) = -\int \frac{y_2 g(x)}{W(y_1, y_2)} dx, \qquad u_2(x) = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx \qquad (44)$$

So, provided we can do these integrals, but the particular solution to the D.E. is $Y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, thus

$$Y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx \qquad (45)$$

Example 30: Solve $y'' + 4y' + 4y = \cosh x$

Solution: The characteristic equation is $r^2 + 4r + 4 = 0 \Rightarrow$ $r = -2, -2$, thus $y_c = (c_1 e^{-2x} + c_2\, x\, e^{-2x})$ & $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$, thus $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = e^{-4x}$, $g(x) = \cosh x$

$$\int \frac{y_2 g(x)}{W(y_1, y_2)} dx = \int \frac{x\, e^{-2x} \cosh x}{e^{-4x}} dx = \int \frac{x\, e^{2x}(e^x + e^{-x})}{2} dx$$

$$= \int \frac{x\,(e^{3x} + e^x)}{2} dx = \frac{1}{2}[x(\frac{e^{3x}}{3} + e^x) - (\frac{e^{3x}}{9} + e^x)]$$

$$\int \frac{y_1 g(x)}{W(y_1, y_2)} dx = \int \frac{e^{-2x} \cosh x}{e^{-4x}} dx = \int \frac{e^{2x}(e^x + e^{-x})}{2} dx$$

$$= \int \frac{(e^{3x} + e^{x})}{2} dx = \frac{1}{2}(\frac{e^{3x}}{3} + e^{x})$$

But

$$Y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$$

$$= -\frac{e^{-2x}}{2} [x(\frac{e^{3x}}{3} + e^{x}) - (\frac{e^{3x}}{9} + e^{x})] + \frac{x\, e^{-2x}}{2} [\frac{e^{3x}}{3} + e^{x}]$$

5.9 <u>Problems</u>

I) Solve the following differential equations

1) $y`` + 2y` + 2y = e^{x} \sin^{2}(2x)$

2) $y`` + y = \sec(x)$

3) $y```+ y``-y`-y = e^{x}$

4) $y`` + y = 1+ \tan x$

5) $y```` + y = 0$

6) $16y``+ 8y` + y = 0$

7) $y`` + 5y` + 6y = 2-x+3x^{2}$

8) $y`` + 3y` + 2y = e^{2x} \cos x$

9) $y`` + 3y` + 2y = \frac{1}{(1+e^x)^2}$

10) $y'' + 3y' - 4y = x\cosh 3x$

11) $y'' + 4y' + 4y = x\sin^2 x$

12) $y'' + 2y' + 4y = x^2$

13) $y'' + 4y = \csc(2x)$

14) $y''' + y'' - y' - y = \cos(2x)$

15) $y'' + 3y' - 4y = e^{5x}$

16) $y'' + 3y' - 4y = e^{-4x} + e^x$

17) $y'' + 5y' + 4y = e^{5x}\cos 2x$

18) $y''' + y'' - y' - y = \sin^2 x$

19) $y'' + 4y' + 5y = \cosh 2x \sin 3x$

20) $2y'' + 5y' + 2y = xe^{4x}$

21) $y'' + 4y = x\cos 3x$

22) $y'' + 4y = 4\tan 2x$

23) $y'' - y = \frac{2}{1+e^x}$

24) $y'' + 9y = \text{cosec}3x$

25) The current *i* flowing through a circuit is given by the equation:

$\frac{d^2i}{dt^2} + 60\ \frac{di}{dt} + 500i = 0$, evaluate the transient current and voltage at time t of the circuit composed of resistance of 20 Ω and inductance with 1 H, given $i(0) = 2$ and $\frac{di}{dt}_{t=0} = 3$.

26) $y``` - 4y`` + y` + 6y = 4 \sin 2x$

27) $y``` - 3y`` + 3y` - y = 3e^x$

28) $y``` - 4y`` - 3y` + 18y = \cos^2 x$

29) $y``` - 6y`` + 11y` - 6y = 2x\,e^{-x}$

30) $y^{(4)} + 2y`` + y = 0$

31) $y^{(4)} - 7y``` + 18y`` - 20y` + 8y = 0$

32) $y^{(4)} + 8y``` + 24y`` + 32y` + 16y = 0$

33) $y^{(4)} + 2y`` + y = \sin 2x$

34) $(D^2 + 9)y = \cos 3x$

35) $y`` - 4y` + 5y = 2\cos x$

36) $y`` + 2y` + 5y = 3\cos(x - \pi/4)$

37) $y`` - 3y` + 2y = 2x^2 + e^x + xe^x + 4e^{3x}$

38) $y`` + 2y` + 5y = e^{-x} \sin 2x$

39) $y'' - 2y' + y = (x^2 - 1)\, e^{2x} + (3x+4)\, e^x$

40) $y'' + y' + 8y = (10x^2 + 21x + 9) \sin 3x + x \cos 3x$

41) $y''' - y' = 4e^{-x} + 3\, e^{2x}$

42) $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$

43) $y'' + y = \cot x$

44) $y'' - 3y' + 2y = \frac{2}{1+e^{-x}}$

45) $y'' - 2y' + y = \frac{e^x}{x}$

46) $(D^2 + 1)y = \sec x \tan x$

47) $(D^2 + 1)y = \csc x$

48) $x^2 y'' + 3x y' - 8y = x$ (put $x = e^t$, $Dy = Ey$, $D^2y = E(E-1)y$)

49) $x^2 y'' - 3x y' + 3y = x^2 \ln x$

50) $y'' + y = \sec^2 x$

51) $y'' - 6y' + 9y = x^{-4} e^{3x}$

52) $y''' - y'' = xe^x + 2x + 1 + 3 \sin x$

53) $y'' + 4y' + 4y = 2x + 3$

II) Solve system of equations:

1) $y' = -6y + 4z$, $z' = -8y + 2z$

2) $y' = -8y - 10z, \quad z' = 5y + 7z, \; y(0) = 1, \; z(0) = 0$

3) $y' = (\sqrt{2}-1)y + (2/\sqrt{3} + 1)z, \quad z' = -\sqrt{3}\, y + (\sqrt{2}+1)z$

III) Choose the correct answer

1) A general solution to $y'' - \sqrt{5}\, y' = 0$ is

(a) $y = C_1 e^{\sqrt{5}t} + C_2 t$

(b) $y = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t$

(c) $y = C_1 e^{\sqrt{5}t} + C_2 t e^{\sqrt{5}t}$

(d) $y = C_1 e^{\sqrt{5}t} + C_2$

2) A suitable form of the general solution to $y'' - 2y' + y = e^t + t$ is

(a) $c_1 te^t + c_2 e^t + At^2 e^t + Bt + C$

(b) $c_1 e^t + c_2 t^2 + Ate^t + Bt + C$

(c) $c_1 te^t + c_2 e^t + Ate^t + Bt + C$

(d) $c_1 te^t + c_2 e^t + Ate^t + Bt$

3) Which one is not the solution of the differential equation $y'' + 3y' + 2y = 0$

$$(e^{-t}, e^{t}, -e^{-t}, -e^{-2t}, 10e^{-t})$$

4) The proper form of particular solution of the differential equation $y'''-3y''+3y'-y = 3e^t$ is

a) Ae^t b) At^2e^t c) At^3e^t d) $(A\cos t + B\sin t)t^2e^t$

e) $(At+B)t^3e^t$

5) The value of the constant r such that $y = x^r$ solves $x^2 y'' + xy'-2y = 0$ for $x > 0$ are

($\pm\sqrt{2}$, $\pm i\sqrt{2}$, $1 \pm\sqrt{2}$, -1 and -2, 1 and -2)